

Automorphism groups of sporadic groups

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Among the simplest invariants of the sporadic finite simple groups are their outer automorphism groups. For 12 of the 26 possible isomorphism types of a sporadic simple group G , the outer automorphism group $\text{Out}(G)$ has order 2, and in the remaining 14 cases, $\text{Out}(G)$ is trivial. Historically the suspicion of the existence of a sporadic group was followed in fairly short order by the calculation of a good upper bound on the size of its outer automorphism group. In a few cases establishing the existence of certain outer automorphisms, like the existence of the groups themselves, presented difficulties overcome only with the use of machine computation. In any case the calculations of the upper bounds, though typically straightforward, can be difficult to track down in the literature – perhaps impossible in some cases. This note, which contains nothing new, is only intended to bring together these calculations. The proximate cause of writing them down was a question from Bob Oliver about the automorphism groups of some of these groups, how they were – or might be – calculated, and specifically whether the Sylow 2-subgroups of a sporadic simple group are self-centralizing in the automorphism group of the simple group. The answer is that they are.

1 Introduction

As each of the 21 twentieth-century sporadic finite simple groups G came into view, the early properties that were worked out included an upper bound on $|\text{Aut}(G)|$, which eventually was proved to be sharp. By comparison with some other invariants – for example, complete local structure, maximal subgroups, and Schur multiplier – the structure of $\text{Aut}(G)$ is relatively easy to settle, except for some nagging cases where one proves that $|\text{Out}(G)| \leq 2$ but equality is a computational challenge. We gather calculations here for the record. In particular they give an affirmative answer to a question of Bob Oliver about sporadic G 's: for $T \in \text{Syl}_2(G)$, is $Z(T)$ a Sylow subgroup of $C_{\text{Aut}(G)}(T)$?

All groups G , X , etc., are to be assumed to be finite. We use the following notation:

G is a finite simple group, identified with its group $\text{Inn}(G)$ of inner automorphisms.

$A = \text{Aut}(G)$ and $\tilde{A} = A/G = \text{Out}(G)$

$\tilde{C}(X) = C_A(X)G/G$ and $\tilde{N}(X) = N_A(X)G/G$, for any $X \subseteq G$

$T \in \text{Syl}_2(G)$, $Z = Z(T)$, and $C = C_G(Z(T))$

M is a (large) proper subgroup of G such that $T_M := T \cap M \in \text{Syl}_2(M)$. A good choice of M reveals much about $\text{Aut}(G)$.

$$F = F^*(M)$$

$\text{Aut}_A(B)$ is the image of the natural mapping $N_A(B) \rightarrow \text{Aut}(B)$ defined by $g \mapsto \text{conjugation by } g$; thus $\text{Aut}_A(B) \cong N_A(B)/C_A(B)$

$m_p(X)$ is the p -rank of the finite group X .

Theorem 1.1. *Let G be one of the sporadic groups listed in Table 1. Then $\tilde{A} = \text{Out}(G)$ is of order 1 or 2, as indicated there. G possesses a subgroup M of the given type. If $\tilde{A} \neq \tilde{1}$, then the next column identifies \tilde{A} and $\tilde{1}$ in terms of M , and the final column gives a 2-subgroup S of G such that $\tilde{C}(S) = 1$.*

We shall prove that $\text{Out}(G)$ has order at most what is listed, and cite the constructions of nontrivial outer automorphisms. Occasionally we get an inductive benefit from our treating all twenty-six groups together.

We make use of the charts of local structure of the sporadic groups found in [10], mainly for centralizers of involutions but occasionally for elements of other small prime orders. The calculations for that information can be made without any use of upper bounds on the size of the automorphism group.

2 Useful Folklore

Lemma 2.1. *Let X be a group and set $Q = F^*(X)$. Then the following conditions hold:*

- (a) $\pi(C_{\text{Aut}(X)}(Q)) \subseteq \pi(F(X))$.
- (b) If Q is a p -group and $P \in \text{Syl}_p(G)$, then $C_{\text{Aut}(X)}(P) = \text{Aut}_{Z(P)}(X)$.
- (c) Suppose that Q is an abelian p -group and $H^1(X/Q, Q) = 1$. Then we have $C_{\text{Aut}(X)}(Q) = \text{Aut}_Q(X)$.
- (d) Suppose that Q is an extraspecial p -group and $X/QX^{(\infty)}$ is a p' -group. Then $C_{\text{Aut}(X)}(Q/Z(Q)) = \text{Aut}_Q(X)$ and $C_{\text{Aut}(X)}(Q) = \text{Aut}_{Z(Q)}(X)$.
- (e) If Q is an abelian or extraspecial p -group and the image of $\text{Aut}_X(Q)$ in $\text{Out}(Q)$ is a self-normalizing subgroup of $\text{Out}(Q)$, then

$$\text{Aut}(X) = \text{Inn}(X)(C_{\text{Aut}(X)}(X/Q) \cap C_{\text{Aut}(X)}(Q)).$$

Proof. In each case we choose some $\alpha \in \text{Aut}(X)$ and argue in the semidirect product

$$H = X \langle \alpha \rangle.$$

In (a), let α be any p -element of $C_X(F^*(X))$ for some prime p , and define $W = \langle \alpha \rangle Z(F(X))$. Then W is abelian. By the F^* -Theorem, $W = \langle \alpha C_X(F^*(X)) \rangle = C_H(F^*(X)) \triangleleft H$. Hence $WF^*(X) = F^*(H)$ and $W = Z(F^*(H))$. As p does not divide $|F(X)|$, $\langle \alpha \rangle = O_p(W) \triangleleft H$, so $\alpha = 1$ as an automorphism of X .

TABLE 1

Pg.	G	$\text{Out}(G)$	M	$\tilde{A}/\tilde{1}$	2-groups
6	M_{11}	1	$M_9 \cong F_{9.8}$		
6	M_{12}	Z_2	M_{11}	$\tilde{A}/\tilde{N}(M)$	$\tilde{C}(T) = 1$
7	M_{22}	Z_2	$M_{21} \cong L_3(4)$	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(T_M) = 1$
8	M_{23}	1	M_{22}		
8	M_{24}	1	M_{23}		
10	J_1	1	$N_G(T)$		
10	$J_2 = HJ$	Z_2	$C = 2_-^{1+4}A_5$	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(O_2(C)) = 1$
10	J_3	Z_2	$C = 2_-^{1+4}A_5$	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(O_2(C)) = 1$
11	J_4	1	$[C, C] = 2_+^{1+12}3M_{22}$		
11	Co_1	1			
12	Co_2	1	$C = 2_+^{1+8}Sp_6(2)$		
12	Co_3	1	$C = 2Sp_6(2)$		
9	HS	Z_2	M_{22}	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(T) = 1$
13	Mc	Z_2	$C = 2A_8$	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(T) = 1$
14	Suz	Z_2	$G_2(4)$	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(T_M) = 1$
15	$He = HHM$	Z_2	E_{5^2}	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(T) = 1$
15	Ly	1	$3Mc2$		
16	Ru	1	${}^2F_4(2)$		
13	$O'N$	Z_2	$C = 4L_3(4)2$	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(T) = 1$
17	Fi_{22}	Z_2	$Z_2 \times U_6(2)$	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(T) = 1$
17	Fi_{23}	1	$2Fi_{22}$		
18	Fi'_{24}	Z_2	Fi_{23}	$\tilde{C}(M)$	$\tilde{C}(T) = 1$
19	$F_5 = HN$	Z_2	A_{12}	$\tilde{N}(M)/\tilde{C}(M)$	$\tilde{C}(T) = 1$
20	$F_3 = Th$	1	$C = 2_+^{1+8}A_9$		
21	$F_2 = BM$	1	$2{}^2E_6(2)$		
22	$F_1 = M$	1	$C = 2_+^{1+24}Co_1$		

For (b) and (c), set $Q = O_p(X)$ and choose any α such that $[\alpha, Q] = 1$. Then $C_H(Q) = Z(Q)\langle\alpha\rangle \triangleleft H$. In particular $[\alpha, X] \leq C_X(Q) = Z(Q)$. The mapping $f : X \rightarrow Z(Q)$ taking $x \mapsto [x, \alpha]$ is a 1-cocycle in $Z^1(X, Z(Q))$. Then f is cohomologically trivial. This holds by hypothesis in (c), and in (b) because f vanishes identically on P and the restriction mapping $H^1(X, Z(Q)) \rightarrow H^1(P, Z(Q))$ is injective, P being a Sylow p -subgroup of X . Therefore there is $z \in Z(Q)$ such that $[x, \alpha] = [x, z]$ for all $x \in X$. Then α is conjugation by x . In (b), the hypothesis implies that $x \in Z(P)$. Thus (b) and (c) hold.

For (d), choose any α such that $[\alpha, Q] \leq Z(Q)$ and set $\bar{H} = H/Z(Q)$. By (a), α is a p -element. Since Q is extraspecial, α induces an inner automorphism on Q , and hence $C_H(Q)$ contains an element $\beta \in Q\alpha$. We set

$$Z_1 = \langle\beta\rangle Z(Q) = \langle\beta\rangle C_X(Q) = C_H(Q) \triangleleft H.$$

We have $[X, Z_1] \leq X \cap Z_1 = Z(Q)$. Let $Y = X^{(\infty)}$. Then $[Y, Z_1, Y] \leq [Z(Q), Y] = 1$ so $[Y, Z_1] = 1$. Therefore $C_H(Z_1)$ contains QY , so by assumption $H/C_H(Z_1)$ is a p' -group. Hence either Z_1 is cyclic or we may write $Z_1 = Z(Q) \times \langle\gamma\rangle$ with $\langle\gamma\rangle \triangleleft H$, for some $\gamma \in Z(Q)\beta \subseteq Q\alpha$. In the latter case $\langle\gamma\rangle \cap X = 1$, so $\gamma \in Z(H)$; in the former case H centralizes $Z_1/\Phi(Z_1)$ so centralizes Z_1 . Thus in any case $\gamma \in Z(H)$, so $\alpha \in QZ(H)$ and the result holds.

Finally in (e), let $A = \text{Aut}(X)$. We have $X/Z(Q) \triangleleft N_A(X)/Z(Q)$, so the self-normalizing hypothesis forces $N_A(X) = XC_A(Q)$, and $[C_A(Q), X] \leq C_X(Q) \leq Q$. The result follows. \square

Corollary 2.2. *Suppose that G is simple of characteristic p -type. Let $P \in \text{Syl}_p(G)$. If $\alpha \in C_{\text{Aut}(G)}(P)$ and α is a nontrivial p' -element, then $C_G(\alpha)$ is strongly p -embedded in G . If $p = 2$, then $C_{\text{Aut}(G)}(P) \leq P$.*

Proof. For any $1 \neq R \leq P$ such $N_P(R) \in \text{Syl}_p(N_G(R))$, $[\alpha, R] = 1$ and so α normalizes $N_G(R)$, centralizing the Sylow p -subgroup $N_P(R)$. Hence $[\alpha, N_G(R)] = 1$ by Lemma 2.1a. This has two consequences: first, normalizers of every extremal subgroup of P lie in $C_G(\alpha)$; second, by Alperin's Theorem, $C_G(\alpha)$ controls strong G -fusion in R . Together these imply that $C_G(\alpha)$ contains $N_G(R)$ for all $1 \neq R \leq P$, proving the first statement. If $p = 2$, then by the Bender-Suzuki theorem, G is a Bender group. In this case it is well-known that $C_{\text{Aut}(G)}(P) \leq P$. \square

Lemma 2.3. *Let G be a simple group and $T \in \text{Syl}_2(G)$. Suppose that*

- (a) $Z := Z(T) = \langle z \rangle \cong Z_2$.
- (b) $F^*(C_G(Z))$ is a 2-group.
- (c) There is $U \triangleleft T$ such that $U \cong E_{2^2}$ and $U^\# \subseteq z^G$.

Let $\alpha \in C_A(T)$. Then one of the groups $W = C_G(\alpha)$, $W = C_G(\alpha z)$ contains $\langle C_G(Z), N_G(U) \rangle$ and satisfies the conditions

- (1) $O_{2'}(W) = Z(W) = 1$, and

(2) $C_G(Z) < W$.

Moreover if $C_G(Z)$ is a maximal subgroup of G , or if G is characterized among all finite groups by the conditions (1) and (2) and the isomorphism type of $C_G(Z)$, then $C_{\text{Aut}(G)}(T) = \text{Aut}_Z(G) \leq \text{Inn}(G)$.

Proof. The importance of such groups U was made clear in the N -group paper of Thompson [32]. Set $T_0 = C_T(U)$, so that $|T : T_0| = 2$ as $U \not\leq Z(T)$. If $m_2(Z(T_0)) > 2$, then $C_{Z(T_0)}(T/T_0)$ is noncyclic and lies in Z , which is a contradiction. So $m_2(Z(T_0)) = 2$ and $U = \Omega_1(Z(T_0)) \text{ char } T_0$. For any $u \in U^\#$, expand T_0 to $T_u \in \text{Syl}_2(C_G(u))$. Since $u \in z^G$, $T_u \in \text{Syl}_2(G)$ and $\text{Aut}_{T_u}(U)$ contains the unique involution of $\text{Aut}(U)$ fixing u . But u was arbitrary, so $\text{Aut}_G(U) \cong \text{Aut}(U)$. By a Frattini argument, there is a 3-element of $N_G(T_U)$ acting nontrivially on U . So $N_G(T_U) \not\leq C_G(Z)$.

Since $F^*(C_G(Z))$ is a 2-group and $Z \leq U$, $F^*(C_G(U)) = F^*(C_{C_G(Z)}(U))$ is a 2-group as well (5.12 of [10]). By Lemma 2.1b applied to both $X = C_G(Z)$ and $X = N_G(T_U)$, α centralizes $C_G(Z)$ and acts on $N_G(T_0)$ as conjugation by 1 or z . Replacing α by αz if necessary, we may assume that $[\alpha, N_G(T_U)] = 1$, whence

$$W := C_G(\alpha) > C_G(Z).$$

If $O_{2'}(W) \neq 1$, then $Y := C_{O_{2'}(W)}(u) \neq 1$ for some $u \in U^\#$. Conjugating by an element of $N_G(U)$, we get $C_{O_{2'}(W)}(z) \neq 1$. But $C_{O_{2'}(W)}(z) \leq O_{2'}(C_W(z)) = O_{2'}(C_G(Z)) = 1$, contradiction. Thus $O_{2'}(W) = 1$. Finally if $Z(W) \neq 1$, it follows that $Z(W) \cap T \neq 1$, so $Z(T) \leq Z(W)$ and $W \leq C_G(Z)$, again a contradiction. Therefore $Z(W) = 1$, completing the proof of (1) and (2). The final statement is an immediate consequence. \square

Lemma 2.4. *Suppose that G acts on the set Ω , and let $\alpha \in \Omega$. Suppose that*

- (a) G acts faithfully and primitively on Ω ;
- (b) G_α acts faithfully and primitively on some G_α -orbit Ψ on $\Omega - \{\alpha\}$;
- (c) Ψ is the unique G_α -orbit of length $|\Psi|$;
- (d) One of the following holds:
 - (1) Conditions (b) and (c) hold for all G_α -orbits on $\Omega - \{\alpha\}$; or
 - (2) For all $\beta \in \Psi$, $C_{\text{Aut}(G_\alpha)}(G_{\alpha\beta}) = 1$; and
- (e) G_α is not cyclic.

Then $C_{\text{Aut}(G)}(G_\alpha) = 1$.

Proof. Let $a \in C_{\text{Aut}(G)}(G_\alpha)$, and set $H = G \langle a \rangle$, the semidirect product. Then H acts on Ω with $H_\alpha = G_\alpha \times \langle a \rangle$. By (c), a stabilizes Ψ . Suppose that $a \downarrow_\Psi \neq 1$. By (b), G_α is faithful and primitive on Ψ . Since $[a, G_\alpha] = 1$, it follows that $\langle a \rangle$ is of prime order and transitive on Ψ . But then G_α embeds in $C_{\Sigma_\Psi}(a \downarrow_\Psi) = \langle a \downarrow_\Psi \rangle$, which contradicts (e). Therefore $a \downarrow_\Psi = 1_\Psi$. If (d1) holds,

then the same argument shows that a fixes Ω pointwise, whence $a = 1$. On the other hand if (d2) holds, then for any $\beta \in \Psi$, a fixes β so a acts on G_β , and (d2) gives $[a, G_\beta] = 1$. In this case $C_G(a)$ contains $\langle G_\alpha, G_\beta \rangle$, which equals G by (a), and again $a = 1$. \square

Lemma 2.5 (cf. [1]). *Suppose that V is an $\mathbf{F}_p G$ -module. Let $x \in G$ be a p' -element such that $C_V(x) = 0$. Put a graph structure on x^G by joining x and y if and only if $\langle x \rangle$ and $\langle y \rangle$ normalize each other. Let C be the connected component of x . If G normalizes C , or if $G = \langle C_G(y) \mid y \in C \rangle$, or if $x \in O_{p'}(G)$, then $H^1(G, V)$ is trivial.*

Proof. It suffices to show that if W is an $\mathbf{F}_p G$ -module containing V with $\dim W = \dim V + 1$ and G centralizing W/V , then $W = V \oplus C_W(G)$. Since x is a p' -element and $C_V(x) = 0$, we have $W = V \oplus W_y$ for each $y \in C$, where we set $W_y = C_W(y)$. Then whenever $y, z \in C$ are connected, z normalizes W_y and so $W_y = W_z$. Hence $W_y = W_x$ for all $y \in C$. Hence W_x is normalized and then centralized by $N_G(C)$, as well as by $\langle N_G(\langle y \rangle) \mid y \in C \rangle$. Obviously $C \subseteq N_G(C)$ so the result follows. \square

Corollary 2.6. *Let $G = F_1$ or Co_2 . Let z be a 2-central involution of G , $C = C_G(z)$, and $Q = O_2(C)$. Then $H^1(C/Q, Q/\langle z \rangle)$ is trivial. Consequently, with Lemma 2.1b, $C_{\text{Aut}(C)}(T) = 1$, for $T \in \text{Syl}_2(G)$.*

Proof. Set $H_z = C_G(z)/O_2(C_G(z))$ and $V_z = O_2(C_G(z))/\langle z \rangle$. In these cases, $(C_G(z)/O_2(C_G(z)), \dim V) = (Co_1, 24)$ or $(Sp_6(2), 8)$ and $V = \Lambda/2\Lambda$ or the spin module, respectively. In either case H_z possesses an element x of order 3 such that $C_V(x)$ is trivial. We let C be the connected component as defined in Lemma 2.4, and $N = N_{H_z}(C)$. By Lemma 2.4t suffices to prove that $N = H_z$.

If $G = Co_2$, then $\dim([x, V_{\text{nat}}]) = 2$ for the natural 6-dimensional module V_{nat} for $Sp_6(2)$. But in that case x has an H_z -conjugate y such that $[x, V_{\text{nat}}] \perp [y, V_{\text{nat}}]$, and then $N \geq \langle C_{H_z}(x), C_{H_z}(y) \rangle \geq \langle E(C_{H_z}(x)), E(C_{H_z}(y)) \rangle = H_z$, as required.

If $G = F_1$, then $H_z \cong Co_1$ and $N_{H_z}(\langle x \rangle) \cong 3Suz2$, and there is $P \in \text{Syl}_3(H_z)$ such that $A := J(P) \cong E_{3^6}$, $x \in A = C_{H_z}(A)$, and $\text{Aut}_{H_z}(A) \cong 2M_{12}$. Moreover, $N_{H_z}(A) \cap N_{H_z}(\langle x \rangle) \cong Z_2 \times M_{11}$. Since $\langle x^{N_{H_z}(A)} \rangle = A$ is abelian, $N_{H_z}(A) \leq N$. Indeed $N_{H_z}(A_0) \leq N$ for any $A_0 \leq A$ such that $C \cap A \neq \emptyset$. Such an A_0 of order 9 exists with $O^2(N_G(A_0)) = O^2(C_G(a)) \cong 3^2U_4(3)$ for every $a \in A_0^\# - C$ [10, Table 5.3]. We fix A_0 . Then for all $a \in A_0^\#$, either $\langle a \rangle \in C$ or $A_0 \triangleleft N_{H_z}(\langle a \rangle)$. Hence setting $\Gamma = \langle N_{H_z}(\langle a \rangle) \mid a \in A_0^\# \rangle$, we have $\Gamma \leq N$. We also fix $a \in A_0 - C$.

We choose any elements $x_i \in O^2(C_{H_z}(a)) \cong 3^2U_4(3)$ of order i for $i = 5, 7$. Then $C_{H_z}(x_i)$ contains A_0 , and hence exactly four elements of C . Therefore x_5 and x_7 are of class $5B$ and $7B$, respectively [10, Table 5.3], and it follows quickly that $N_{H_z}(\langle x \rangle_i) \leq \Gamma \leq N$. Next, let $t \in H_z$ be any 2-central involution centralizing any element $c \in C$; such elements exist, with $C_{H_z}(ct)$ an extension of 2_-^{1+6} by $\Omega_6^-(2)$ and $C_{H_z}(t)$ an extension of 2_+^{1+8} by $\Omega_8^+(2)$. Thus there is a

conjugate $c' \in c^{C_{H_z}(t)}$ of c such that $[c, c'] = 1$ and $C_{H_z}(t) \leq \langle C_{H_z}(c), C_{H_z}(c') \rangle$. Hence $C_{H_z}(t) \leq \Gamma \leq N$.

Now $N \geq N_{H_z}(X)$ for $X = A, \langle x_5 \rangle, \langle x_7 \rangle$, and $\langle t \rangle$, as well as $\langle c \rangle$ for any $c \in C$. Together these imply that $|H_z : N|$ divides $5 \cdot 23$. But also $|H_z : N| \equiv 1 \pmod{33}$, since $N \geq N(A) \geq N(P)$ and Sylow 11-normalizers of $N_G \langle x \rangle \leq N$ are Sylow 11-normalizers in G . Therefore $|H_z : N| = 1$ and the proof is complete. \square

3 Group by Group Calculations

G \cong **M**₁₁ [5]

Since G is sharply quadruply transitive on 11 letters, the stabilizer $G_{\alpha\beta}$ of two points is sharply doubly transitive – i.e., a Frobenius group – of order 9.8. Then $M := N_G(G_{\alpha\beta})$ satisfies $|M| = 2|G_\alpha|$ and M is easily seen to be a Sylow 3-normalizer in G . Since $|G : M|$ has no nontrivial divisors that are congruent to 1 (mod 3), M is maximal in G . By a Frattini argument,

$$\tilde{A} = \tilde{N}(M).$$

M is complete ($\text{Aut}(M) = \text{Inn}(M) \cong M$) so $N_G(M) = MC_G(M)$ and

$$\tilde{N}(M) = \tilde{C}(M).$$

Since $T \leq M$, $C_A(M) \leq C_A(T)$. Let $\alpha \in C_A(M)$. Then α maps into $C_{\text{Aut}(C)}(T)$. But $C_G(Z) \cong GL_2(3)$. By Lemma 2.1b, α centralizes $C_G(Z)$. Since M is maximal in G , $G = \langle C_G(Z), M \rangle$ is centralized by α , so $\alpha = 1$, proving that

$$\tilde{C}(M) = 1. \quad \square$$

G \cong **M**₁₂ [5]

We show first that G has at most two conjugacy classes of M_{11} -subgroups.

Suppose that H and K are nonconjugate M_{11} -subgroups of G . Then K has no fixed point on G/H . Since $|G : H| = |G : K| = 12$ and K has an element of order 11, $HK = G$. Therefore $|K : H \cap K| = 12$. Consequently H and K share no Sylow 3-subgroup. We have proved that M_{11} -subgroups of G sharing a Sylow 3-subgroup are conjugate, as are any two M_{11} -subgroups whose product is not G .

Now H has a Sylow 3-subgroup of order 3^2 , all of whose nonidentity elements are fused in H . But in G , which has a Sylow 3-subgroup of type 3^{1+2} , there are at most 2 (exactly two, in fact) conjugacy classes of subgroups of order 3^2 all of whose nonidentity elements are fused in G . By the previous paragraph, there are at most two conjugacy classes of M_{11} subgroups in G . Thus

$$|\tilde{A} : \tilde{N}(M)| \leq 2.$$

Since $M \cong M_{11}$ and we have just seen that $\text{Out}(M_{11}) = 1$,

$$\tilde{N}(M) = \tilde{C}(M).$$

M is doubly transitive, hence primitive, on $G/M - \{M\}$. By Lemma 2.4(d1),

$$\tilde{C}(M) = 1.$$

Finally, for any $\alpha \in \tilde{C}(T)$, $T_M \leq M \cap M^\alpha$, whence $|M : M \cap M^\alpha|$ is odd. By the second paragraph, M^α is G -conjugate to M . Thus, $\alpha \in GN_A(M)$, and so

$$\tilde{C}(T) \leq \tilde{N}(M) = \tilde{C}(M) = \tilde{1}. \quad \square$$

Existence note: A subgroup $H = M_{12}.2$ with $F^*(H) = M_{12}$ is visible in M_{24} as the stabilizer of a decomposition of the underlying 24-element set into two complementary dodecads of the Golay code, proving that the bound $|\tilde{A}| \leq 2$ is sharp and exhibiting both of the quasi-equivalent transitive actions of M_{12} on 12 points [4].

G \cong M₂₂ [5]

Since M_{24} is quintuply transitive on 24 points and preserves a Steiner system $S(5, 8, 24)$, its three-point stabilizer M_{21} is doubly transitive on 22 points and preserves a Steiner system $S(2, 5, 21)$, i.e. the projective plane of order 4.

The maximal parabolic subgroup $P = M_{20} \cong ASL_2(4)$ of $M_{21} \cong L_3(4)$ has no faithful permutation representation of degree less than 16. (A point stabilizer P_α would satisfy $1 < O_2(P) \cap P_\alpha < O_2(P)$ so would be reducible on $O_2(P)$, contain no element of order 5, then be of index 10; but 3-elements of P leave no hyperplane of $O_2(P)$ invariant.) As a result, if M_{21} acts on any set of cardinality 22, it must be transitive or have a fixed point; the latter must in fact hold since 11 does not divide $|M_{21}|$. Consequently all M_{21} -subgroups of M_{22} are M_{22} -conjugate and

$$\tilde{A} = \tilde{N}(M).$$

As $\text{Out}(M)$ is rather large – of order 12 – we bring other subgroups to bear as well. M has exactly two conjugacy classes of (maximal parabolic) subgroups isomorphic to 2^4A_5 , represented by N_0 and N_1 , say, and such that if we put $Q_i = O_2(N_i)$, then $N_G(Q_0)/Q_0 \cong A_6$ and $N_G(Q_1)/Q_1 \cong \Sigma_5$. Therefore Q_0 and Q_1 are not A -conjugate. Thus if we set $N_{Q_0} = N_A(M) \cap N_A(Q_0)$, we have

$$\tilde{N}(M) = \tilde{N}_{Q_0} \text{ and } \text{Aut}_G(Q_0) \cong A_6.$$

Set $C_{Q_0} = C_{N_{Q_0}}(Q_0)$ and $B = \text{Aut}_G(Q_0)\text{Aut}_{N_{Q_0}}(Q_0)$. As $G \triangleleft A$, we have $\text{Aut}_G(Q_0) \triangleleft B$. This forces¹ $|B : \text{Aut}_G(Q_0)| \leq 2$. Moreover $B/\text{Aut}_G(Q_0) \cong \tilde{N}_{Q_0}/\tilde{C}_{Q_0}$, so

$$|\tilde{N}_{Q_0}/\tilde{C}_{Q_0}| \leq 2.$$

¹ $\text{Aut}(Q_0) \cong GL_4(2) \cong A_8$. Then $\text{Aut}(Q_0)$ and $\text{Aut}_G(Q_0) \cong A_6$ share a Sylow 3-subgroup P . One calculates $|N_{\text{Aut}_G(Q_0)}(P) : P| = 4$ and $|N_{\text{Aut}(Q_0)}(P) : P| = 8$. By a Frattini argument $B \leq \text{Aut}_G(Q_0)N_{\text{Aut}(Q_0)}(P)$, so $|B : \text{Aut}_G(Q_0)| \leq |N_{\text{Aut}(Q_0)}(P) : N_{\text{Aut}_G(Q_0)}(P)| = 2$.

But $\text{Aut}(M) \cong \text{P}\Gamma\text{L}_3(4) \langle \tau \rangle$, where $\tau : D \mapsto (D^T)^{-1}$ for projective images D of 3×3 matrices. By a direct calculation, we see that $C_{\text{Aut}(M)}(Q_0) = 1$, so

$$\tilde{C}_{Q_0} = \tilde{C}(M).$$

M is doubly transitive, hence primitive, on $G/M - \{M\}$. By Lemma 2.4 (condition (d1) holds),

$$\tilde{C}(M) = 1.$$

Finally, let $\beta \in C_A(T)$. Then β centralizes Q_0 and the other E_{2^4} -subgroup Q_1 of $T_M \in \text{Syl}_2(M)$. We have $Q_i = F^*(N_G(Q_i))$, $i = 0, 1$, so β centralizes $N_G(Q_i)/Q_i$. Thus β normalizes every overgroup of Q_i in $N_G(Q_i)$. In particular β normalizes $\langle N_M(Q_0), N_M(Q_1) \rangle = M$. Thus $\beta \in C_{Q_0}$ and

$$\tilde{C}(T) = \tilde{C}_{Q_0} = \tilde{C}(M) = 1. \quad \square$$

Existence note: The setwise stabilizer of a two-point set in the natural action of M_{24} on 24 letters is a group $H = M_{22}.2$ with $F^*(H) = M_{22}$, so the bound $|\text{Out}(M_{22})| \leq 2$ is sharp.

Since $\tilde{A} = \tilde{N}_{Q_0}$, $N_{\text{Aut}(M_{22})}(M) = M \langle \sigma \rangle$, where σ is a field automorphism of $M \cong L_3(4)$. Hence $\text{Aut}_{M_{22}}(Q_1) = \text{Aut}_{\text{Aut}(M_{22})}(Q_1)$ and $O_2(N_{\text{Aut}(M_{22})}(Q_1)) \cong E_{2^5}$. On the other hand as $\tilde{N}_{Q_0}/\tilde{C}_{Q_0}$ has order 2, $\text{Aut}_{\text{Aut}(M_{22})}(Q_0) \cong N_{A_8}(A_6) \cong \Sigma_6$.

$\mathbf{G} \cong \mathbf{M}_{23} \quad [5]$

Since a Sylow 2-subgroup of M_{21} has no faithful permutation representation of degree less than 16, the same is true for M_{22} . Nor can M_{22} , by its order, act transitively on 23 letters. Therefore as in the M_{22} argument above, any subgroup of G isomorphic to $M \cong M_{22}$ has a fixed point on G/M , so is conjugate to M , and

$$\tilde{A} = \tilde{N}(M).$$

Let $Q_i \leq M$, $i = 0, 1$, be the same subgroups as in the M_{22} analysis, so that $N_i := N_M(Q_i)$ satisfies $F^*(N_i) = Q_i$, $N_0/Q_0 \cong A_6$, and $N_1/Q_1 \cong \Sigma_5$. Then $N_i^* := N_{M_{23}}(Q_i)$ satisfies $F^*(N_i^*) = Q_i$ and $\text{Aut}_{M_{23}}(Q_i) \cong A_7$ or Σ_6 according as $i = 0$ or 1 . Since any subgroup of $\text{Aut}(Q_0) \cong A_8$ isomorphic to A_7 is self-normalizing in A_8 , and $\text{Aut}_{M_{23}}(Q_0) \triangleleft \text{Aut}_A(Q_0)$, we have $\text{Aut}_A(Q_0) = \text{Aut}_G(Q_0)$. But we saw in the M_{22} analysis that $\text{Aut}_{\text{Aut}(M_{22})}(Q_0) \cong \Sigma_6$, which does not embed in A_7 . Consequently $\text{Aut}_A(M) = \text{Inn}(M)$, so

$$\tilde{N}(M) = \tilde{C}(M).$$

M is doubly transitive, hence primitive, on $G/M - \{M\}$. By Lemma 2.4 (condition (d1) holds),

$$\tilde{C}(M) = 1. \quad \square$$

$\mathbf{G} \cong \mathbf{M}_{24} \quad [5]$

Let $\alpha \in A$ and $H = M^\alpha$, and suppose by way of contradiction that H is not G -conjugate to M . Fix a series $G > M = M_{23} > M_{22} > M_{21} > M_{20}$ and let $H_n = H \cap M_n$, $n = 20, 21, 22, 23$. We claim that $|M_n : H_n| = 24$ for each such n . It is sufficient to show by descending induction that H_n is transitive on M_n/M_{n-1} for $n \geq 21$. Since H has no fixed points on G/M and contains an element of order 23, the claim holds for $n = 24$. For other values of n , the orbit lengths of H_n on M_n/M_{n-1} are restricted by the conditions H_n , $n \geq 21$, contains elements g_p of orders $p = 5, 7, 11$, and 23, with 4, 3, 2, and 1 fixed points respectively on G/M . Thus, g_{23} implies transitivity for $n = 23$; g_{11} and g_7 for $n = 22$; and g_7 and g_5 for $n = 21$. Now $M_{20} \cong ASL_2(4)$ and $|M_{20} : H_{20}| = 24$. But no such subgroup H_{20} of M_{20} exists; we would have $H_{20} \cap O_2(M_{20}) \neq 1$ and 5 divides $|H_{20}|$, whence $O_2(M_{20}) \leq H_{20}$ and $|M_{20} : H_{20}| = 12$, contradiction.

Therefore H is G -conjugate to M , and

$$\tilde{A} = \tilde{N}(M).$$

Since $\text{Out}(M) = 1$,

$$\tilde{N}(M) = \tilde{C}(M).$$

M is doubly transitive, hence primitive, on $G/M - \{M\}$. By Lemma 2.4 (condition (d1) holds),

$$\tilde{C}(M) = 1. \quad \square$$

G \cong **HS** [18]

Let $\alpha \in \text{Aut}(G)$, set $H = M^\alpha$, and assume that H is not G -conjugate to M .

G is a rank 3 permutation group on M , and the M -orbits on $G/M - \{M\}$ are of orders 22 and 77. Since $1 + 22$ does not divide $|G : M| = 100$, this action of G on 100 points is primitive. Elements of H of order 7 or 11 have G -conjugates $g_7, g_{11} \in M$ by Sylow's Theorem. Then g_7 has one fixed point on the 22-orbit and none on the 77-orbit; g_{11} has no fixed points on either of these orbits. Hence g_7 and g_{11} fix two and one points, respectively, of G/M . Therefore there is an H -orbit Ψ on G/M such that $|\Psi| \equiv 1 \pmod{11}$. Since $|\Psi|$ divides $|H|$ and $|\Psi| \equiv 0, 1, \text{ or } 2 \pmod{7}$, the only possibility is $|\Psi| = 56$.

However, M_{22} has no transitive action of degree 56. For if it did, then a point stabilizer K in M_{22} would have order $2^4 \cdot 3^2 \cdot 5 \cdot 11$ and contain a Sylow p -normalizer of M_{22} for both $p = 5$ and $p = 11$. Hence for any $g \in M_{22} - K$, $K_0 := K \cap K^g$ would have order dividing, and then equaling, $2^4 \cdot 3^2$. No 3-local subgroup of M_{22} contains a group of order 16, so $F^*(K_0) = O_2(K_0)$, and the only possibility admitting a group of order 9 is $O_2(K_0) \cong E_{2^4}$. Therefore K , and $K \cap M_{21}$, would have elementary abelian Sylow 2-subgroups. But since 11 divides $|K|$, $|K \cap M_{21}| = 2^a \cdot 3^2 \cdot 5$, where $a = 3$ or 4. No such subgroup of M_{21} exists apart from A_6 , which has nonabelian Sylow 2-subgroups.

Therefore we have a contradiction, and

$$\tilde{A} = \tilde{N}(M).$$

As $M \cong M_{22}$,

$$\tilde{N}(M)/\tilde{C}(M) \leq \text{Out}(M) \cong Z_2.$$

We have seen that $C_{\text{Aut}(M_{22})}(M_{21}) = 1$, and the action of M on the 22-orbit is primitive. Hence condition (d2) of Lemma 2.4 holds, and so

$$\tilde{C}(M) = 1.$$

Finally let $\beta \in C_A(T)$. Let $Q_0, Q_1 \leq T_M$ be as in the M_{22} -analysis. Since $m_2(G) = 4$ (visible from the $4^3L_3(2)$ 2-local subgroup), and $N_G(Q_i)$, like $N_M(Q_i)$, is irreducible on Q_i , we have $Q_i = \Omega_1(O_2(N_G(Q_i)))$. Since $N_M(Q_i)$ contains an element of order 5, either $O_2(N_G(Q_i)) = Q_i$ or $|O_2(N_G(Q_i))| \geq 2^8$. But $|T| = 2^9$ and $|N_M(Q_i)/Q_i| = 2^3$, so $Q_i = O_2(N_G(Q_i))$. By Lemma 2.1e, α centralizes $N_G(Q_i)/Q_i$, so normalizes $N_M(Q_i)$, $i = 0, 1$. As $M = \langle N_M(Q_0), N_M(Q_1) \rangle$, α normalizes M . By the M_{22} analysis, $[\alpha, M] = 1$. Hence $C_G(\alpha) \geq \langle M, T \rangle = G$, $\alpha = 1$, and

$$\tilde{C}(T) = 1. \quad \square$$

Existence note: The original construction of HS [18] constructed a group X with $F^*(X) = HS$ and $|X : F^*(X)| = 2$. The full automorphism group also appears as $C_{F_5}(t)/\langle t \rangle$ for certain involutions $t \in F_5$. (The group $O_2(C_{\text{Aut}(F_5)}(t))$ is cyclic of order 4.)

$\mathbf{G} \cong \mathbf{J}_1 \quad [20]$

As $T \in \text{Syl}_2(G)$, a Frattini argument gives

$$\tilde{A} = \tilde{N}(M).$$

Since $M = N_G(T)$ is complete,

$$\tilde{N}(M) = \tilde{C}(M).$$

Let $\alpha \in C_A(M)$. For any involution $x \in M$, $C_M(x) \cong Z_2 \times A_4$, so the action of α on $C_G(x) \cong Z_2 \times A_5$ lies in $C_{\text{Aut}(Z_2 \times A_5)}(Z_2 \times A_4) = 1$. Therefore α centralizes $H := \langle N_G(T), C_G(x) \mid x \in T^\# \rangle$. By the Bender-Suzuki Theorem, G has no strongly embedded subgroup. But H is strongly embedded in G unless $H = G$, which must therefore be the case. Then $\alpha = 1$ and

$$\tilde{C}(M) = 1. \quad \square$$

$\mathbf{G} \cong \mathbf{J}_2, \mathbf{J}_3 \quad [21, 13, 19]$

For both $G = J_2$ and $G = J_3$, $M = C_G(Z) \cong 2_-^{1+4}\Omega_4^-(2) = C_G(Z)$. As Z is a Sylow center,

$$\tilde{A} = \tilde{N}(M).$$

We have $F = F^*(M) = O_2(M)$. By Lemma 2.1d, $N_A(M) \cap C_A(F) = C_A(M)$. Therefore as $\text{Out}(F) \cong O_4^-(2)$,

$$\tilde{N}(M)/\tilde{C}(M) \leq \text{Out}(F)/\Omega_4^-(2) \cong Z_2.$$

Finally, all subgroups of F of order 2 are G -conjugate to Z . By a standard result on p -groups (10.11 of [9]), there is $U \cong E_{2^2}$ such that $U \leq F$ and $U \triangleleft T$. Thus Lemma 2.3 applies to give its conclusions (1) and (2). Moreover, in [21], Z. Janko characterized the orders of the two groups J_2 and J_3 (and much more) by those conclusions (1) and (2), and the structure of $C(Z)$. Although two groups emerge, neither of J_2 and J_3 contains the other, by Lagrange's Theorem. Hence $W = G$ in the language of Lemma 2.3, and so

$$\tilde{C}(M) \leq \tilde{C}(T) = 1. \quad \square$$

Existence note: Constructions of J_2 , e.g. as a rank 3 permutation group [13], and of J_3 [19], construct a non-inner automorphism as well. $\text{Aut}(J_2)$ is visible in a number of groups, for example as the fixed points of a non-inner automorphism of Suz . As a pariah, J_3 is not obviously on view elsewhere, and as proved by Griess is not involved in F_1 .

$\mathbf{G} \cong \mathbf{J}_4 \quad [22]$

G is of characteristic 2 type, and $Z(T)$ is cyclic. Therefore as $M = [C, C]$,

$$\tilde{A} = \tilde{N}(C) = \tilde{N}(M).$$

$O_2(C)/Z \cong E_{2^{12}}$ is an absolutely irreducible (faithful) 6-dimensional module for $O_2^2(C)/O_2(C) \cong 3M_{22}$ over \mathbf{F}_4 . Therefore $\text{Aut}_C(O_2(C))$ is self-normalizing in $\text{Aut}(O_2(C))$. By Lemma 2.1e,d applied to C , M , respectively,

$$\tilde{N}(C) \leq \tilde{C}(M).$$

Let $\alpha \in C(M)$ be of prime order p . If p is odd then $[\alpha, C] = 1$ by Lemma 2.1a and then Corollary 2.2 gives a contradiction. So $p = 2$. Choose $x \in C$ of order 11 and let $N = N_G(\langle x \rangle)$. Then $R := F^*(N) \cong 11^{1+2}$ and $N \cap O_{2,3}(M) \cong SL_2(3)$ is absolutely irreducible on R . Therefore α centralizes or inverts $R/\Phi(R)$. Replacing α by αz if necessary we may assume that $[\alpha, R] = 1$, whence $N \leq H := C_G(\alpha)$ by Lemma 2.1a. As $M \leq H$, we now have $|G : H|$ dividing 2.23.29.31.37.43. Moreover by Corollary 2.2, H is strongly 11-embedded in G , whence $|G : H| \equiv 1 \pmod{11^3}$. These conditions imply that $|G : H| = 1$, so

$$\tilde{C}(M) = 1. \quad \square$$

$\mathbf{G} \cong \mathbf{Co}_1 \quad [4]$

The action of $G = Co_1$ on the Leech lattice Λ gives an absolutely irreducible embedding $G \leq PSL_{24}(\mathbf{Z}) \leq PGL_{24}(\mathbf{Q})$. We identify Co_1 with its image. This is the only complex irreducible projective representation of G of degree at most 24. Therefore any automorphism $\alpha \in A$ is realized as conjugation by some element $a \in N_{GL_{24}(\mathbf{Q})}(G)$. Let $V = \mathbf{Q}^{24}$, and let $\Lambda \subseteq V$ be a copy of the Leech lattice invariant under $Co_0 = 2Co_1$. Then $\mathbf{Q}\Lambda = V$, and we may consider $a \in GL(V)$. Hence $\Lambda^a \subseteq V$, and $\Lambda/\Lambda^a \cap \Lambda$ is finite and Co_0 -invariant. However, the representation of Co_0 on Λ has the property that it remains irreducible (mod p) for every prime p , i.e., Co_0 is irreducible on $\Lambda/p\Lambda$ for every prime p . It follows that $\Lambda^a \cap \Lambda = n\Lambda$ for some integer n , and similarly $\Lambda^a \cap \Lambda = m\Lambda^a$ for some m . Therefore modifying a by a rational scalar we may achieve $\Lambda^a = \Lambda$. By absolute irreducibility there is a unique bilinear symmetric nondegenerate rational G -invariant form f on Λ , up to scalars, and there is a unique one which is unimodular on Λ . Therefore a preserves f , whence $a \in \text{Aut}(\Lambda)$, which by definition is Co_0 . Therefore

$$\tilde{A} = 1. \quad \square$$

G \cong Co₂ [4]

The action of $M/O_2(M) \cong Sp_6(2)$ makes the Frattini quotient of $O_2(M)$ the spin module, which is absolutely irreducible. By Corollary 2.6 and Lemma 2.1b,

$$\tilde{A} = \tilde{N}(M) = \tilde{C}(M) = \tilde{C}(T),$$

It remains to show that $\tilde{C}(M) = 1$. Let $\alpha \in C(M)$ be of prime order p . As G is of characteristic 2 type, $p = 2$ by Corollary 2.2. Let $t \in T$ be an involution in class $2B$, and extremal in T . Then $F^*(C_G(t)) = O_2(C_G(t))$, and $Z(O_2(C_G(t)))$ is the direct product of a trivial module and a natural module for $C_G(t)/O_2(C_G(t)) \cong A_8$. Hence $Z(C_T(t)) = \langle t \rangle Z$. By Lemma 2.1b, $\alpha \downarrow_{C_G(t)} \in \text{Aut}_{Z(C_T(t))}(C_G(t)) = \text{Aut}_{\langle z \rangle}(C_G(t))$. Replacing α by αz if necessary we may assume that $H := C_G(\alpha)$ contains $\langle C, C_G(t) \rangle$. The subgroups of C and $C_G(t)$ of order 5 are not G -conjugate so a Sylow 5-subgroup F of H is not cyclic. In particular some $x \in H$ of order 5 is 5-central in G . But $R := F^*(C_G(x)) = O_5(C_G(x)) \cong 5^{1+2}$. Since $[\alpha, F] = 1$ and α has order 2, $[\alpha, R] = 1$ and then $N_G(\langle x \rangle) \leq H$. Thus, $|G : H|$ divides $3^4 \cdot 11 \cdot 23$ and $|G : H| \equiv 1 \pmod{5^2 \cdot 7}$. The last holds because C is strongly 7-embedded in G and because $\langle x \rangle$ is weakly closed with $N_G(\langle x \rangle) \leq H$, with $R \cong 5^{1+2}$. But there are no such numbers $|G : H| > 1$, so $H = G$ and

$$\tilde{C}(M) = 1. \quad \square$$

G \cong Co₃ [4]

In Co_3 there are two classes of involutions, with centralizers $M = 2Sp_6(2)$ and $M_1 = Z_2 \times M_{12}$, which we may assume are chosen so that $T \in \text{Syl}_2(M)$ and $TY_1 := T \cap M_1 \in \text{Syl}_2(M_1)$. In particular there is a unique class of 2-central

involutions. Since $Sp_6(2)$ is complete,

$$\tilde{A} = \tilde{N}(M) = \tilde{C}(M).$$

Let $\alpha \in C(T)$. Then α normalizes $M = C_G(Z)$, and so acts on M like an element of $Z(T)$, as T/Z is self-centralizing in the *Chev*(2)-group M/Z . But $Z(T) = Z \leq Z(M)$, so $C(T) = C(M)$.

Now α acts on M_1 and α centralizes T_1 . The restriction of α to M_1 therefore lies in $C_{\text{Aut}(M_1)}(T_1) = \text{Aut}_Z(M_1)$, as we have seen in the M_{12} -case. Replacing α by αz , where $Z = \langle z \rangle$, we may assume that $[\alpha, M_1] = 1$. Set $H = C_G(\alpha)$. Thus $\langle M, M_1 \rangle \leq H$. There is $R \leq G$ such that $R = C_G(R) \cong E_{3^5}$ and $N_G(R) = RL$, $L \cong Z_2 \times M_{11}$. Replacing R by a conjugate we may assume that $L \leq M_1$. Then any involution $z \in L$ is M_1 -conjugate to z , so $H \geq \langle C_G(t) \mid t \in L, t^2 = 1 \neq t \rangle \geq R$. Also M and M_1 have non-conjugate Sylow 5-subgroups so 5^2 divides $|H|$. As $|M|$, $|M_1|$, and $|RL|$ divide $|H|$, $|G : H|$ divides $5 \cdot 23 = 115$. But $R = J(P)$ for $P \in \text{Syl}_3(G)$, so $|G : H| \equiv 1 \pmod{3}$. If $\alpha \neq 1$ it follows that $|G : H| = 5 \cdot 23$. But α acts on $W := O_5(C_G(x))$ for $x \in M$ of order 5, and $W \cong 5^{1+2}$, with $|C_W(\alpha)| = |H \cap W| = 5^2$, $C_G(W) = Z(W)$, and $N_G(W)/W$ of order 48 and irreducible on $W/Z(W)$. On the one hand, $|C_W(\alpha)| = |H \cap W| = 5^2$ forces α to be a 5-element. Then on the other hand, α centralizes $N_G(W)/W$, which is irreducible on $W/Z(W)$ and so cannot normalize $C_W(\alpha)$, a contradiction. Therefore $\alpha = 1$ and so

$$\tilde{C}(M) = 1 = \tilde{C}(T). \quad \square$$

G \cong **Mc** [25]

$M = C_G(Z)$ so

$$\tilde{A} = \tilde{N}(M).$$

$M \cong 2A_8$ so

$$|\tilde{N}(M)/\tilde{C}(M)| \leq |\text{Out}(M)| = 2.$$

Let $\alpha \in C_A(M)$. Choose any $Q \leq T$ such that $Q \cong E_{2^4}$ and $N_T(Q) \in \text{Syl}_2(N_G(Q))$. Then α normalizes Q and $N_G(Q)$, and $\text{Aut}_G(Q) \cong A_7$. Since A_7 is maximal in $\text{Aut}(Q) \cong A_8$, and since $H^1(A_7, Q) = 1$, Lemma 2.1c implies that $[\alpha, N_G(Q)] = 1$. Therefore $C_G(\alpha)$ contains $\langle M, N_G(Q) \rangle > M$. But $\langle M, N_G(Q) \rangle = G$, so $\alpha = 1$ and

$$\tilde{C}(M) = 1. \quad \square$$

Existence note: $\text{Aut}(Mc)$ is visible in Co_1 as the stabilizer $\cdot 223$ of a certain triangle in the Leech lattice [6]. It is also visible in Ly as $N_{Ly}(V)/V$ for a certain subgroup $V \leq Ly$ of order 3 [24].

G \cong **O'N** [26, 30, 2, 23]

Since $M = C_G(Z)$,

$$\tilde{A} = \tilde{N}(M).$$

Also $|\text{Out}(M)| = 4$ and $|\text{Aut}(M) : \text{Aut}_G(M)| = 2$, so

$$\tilde{N}(M)/\tilde{C}(M) \leq \text{Out}(M)/\text{Out}_G(M) \cong Z_2.$$

We have $T \leq M$. Let $\alpha \in C_A(M) \leq C_A(T)$ and set $C_\alpha = C_G(\alpha)$. There is $M_1 \leq G$ such that $T \leq M_1$ and $F^*(M_1) \cong Z_4 \times Z_4 \times Z_4$. By Lemma 2.1b, $[\alpha, M_1] = 1$.

We claim that $C_G(t) \leq C_\alpha$ for every involution $t \in C_\alpha$. To do this it is enough to consider only those $t \in T$. Since $M_1 = [M_1, M_1]$, we have $O^2(C_\alpha) = C_\alpha$, and so by the Thompson Transfer Lemma (15.16 of [10]), t has a C_α -conjugate in $E(C) \cong 4L_3(4)$. Then as $L_3(4)$ has only one class of involutions, t is C_α -conjugate into $O_2(M_1)$, and then C_α -conjugate to $z \in Z$. As $C_G(z) \leq C_\alpha$, the claim is proved. By the Bender-Suzuki Theorem, $C_\alpha = G$, so $\alpha = 1$ and we have proved

$$\tilde{C}(T) \leq \tilde{C}(M) = 1. \quad \square$$

Existence note: Sims [30] constructed $O'N$ together with a non-inner automorphism. See also [2, 23].

G \cong **Suz** [31]

G is a rank 3 permutation group of degree $1782 = 2 \cdot 3^4 \cdot 11$ with point stabilizer $M \cong G_2(4)$. The subdegrees (M -orbitlengths) are $1 + 1365 + 416$, with the two-point stabilizer corresponding to the 416 suborbit being isomorphic to J_2 . The 1365-orbit stabilizer in M is a maximal parabolic subgroup P of M . This action of G is primitive as 417 does not divide 1782.

$T_M := M \cap T = C_T(U)$ where U is the unique² normal four-subgroup of T . $N_M(U) = O^2(N_G(U))$. So $A = GN_A(T)$ and $N_G(T)$ normalizes $N_G(U)$ and $N_M(U)$. Let $\beta \in N_A(U)$. If $M^\beta \neq M$, then $M \cap M^\beta = N_M(U)$, which is a maximal parabolic subgroup of M . Hence M^β has an orbit of length $|M : N_M(U)| = 1365$ on G/M . The remaining 417 points fall into one or two orbits since G has rank 3 on both G/M and G/M^β . The length of any nontrivial orbit of M^β of odd degree is a multiple of the index 1365 of any maximal parabolic subgroup of M , by Tits's Lemma. Therefore M^β has a fixed point on G/M , and so is G -conjugate to M . So

$$\tilde{A} = \tilde{N}(M).$$

Obviously

$$\tilde{N}(M)/\tilde{C}(M) \leq \text{Out}(G_2(4)) \cong Z_2.$$

$\text{Out}(M)$ is generated by the image of a field automorphism, and it follows easily that $C_{\text{Aut}(M)}(P) = 1$. Hence Lemma 2.4 applies, with condition (d2) holding, and we conclude that

$$\tilde{C}(M) = 1.$$

²Uniqueness is visible in $C_G(Z) \cong 2_-^{1+6} O_6^-(2)$.

Finally, we claim that the two maximal parabolic subgroups P and P_1 of M containing T_U are $C_A(T)$ -invariant. We have $P = N_M(U)$ with U the unique normal four-subgroup of T , and by a standard commutator computation in T_U , $P_1 = N_M(Q_1)$ where $Q_1 = Z_2(T_U)$. As $T_U = C_T(U)$, both $X := N_G(U)$ and $X_1 := N_G(Q_1)$ contain T . Also $Z \leq U \cap Q_1$, so since $F^*(C) = O_2(C)$, it follows that $Q_0 := F^*(X) = O_2(X)$ and $Q_1 := F^*(X_1) = O_2(X_1)$. Therefore by Lemma 2.1e, α normalizes any overgroup of Q_0 in X as well as any overgroup of Q_1 in X_1 . Hence to prove the claim, it is enough to show that $Q_i \leq T_U$, $i = 0, 1$, for then $Q_0 \leq P$ and $Q_1 \leq P_1$. Now for $i = 0$ and i , $Z \leq Z(T) \leq Z(Q_i)$ and for a Cartan subgroup H of $N_M(T_U)$, $U = \langle Z^H \rangle \leq Z(Q_i)$, whence $Q_i \leq T_U$, as desired, proving the claim. Therefore $C_A(T)$ normalizes $\langle P, P_1 \rangle = M$, and then acts on M as a subgroup of $C_{\text{Aut}(M)}(T_U) = \text{Aut}_U(M)$. Hence

$$\tilde{C}(T) \leq \tilde{C}(M) = 1. \quad \square$$

Existence note: In $H = Co_1$, there is a subgroup Y of order 3 such that $F^*(Y) \cong 3\text{Suz}$ and $|Y : F^*(Y)| = 2$, so the bound $|A| \leq 2$ is sharp.

G \cong **He** [16, 17]

Here $M \in \text{Syl}_5(G)$ with $M \cong E_{5^2}$. By a Frattini argument

$$\tilde{A} = \tilde{N}(M).$$

Moreover $\text{Aut}_G(M) \cong SL_2(3) * Z_4$, so that $Z(\text{Aut}(M)) \leq \text{Aut}_G(M)$ and the image I of $\text{Aut}_G(M)$ in $\text{Aut}(M)/Z(\text{Aut}(M)) \cong PGL_2(5)$ is isomorphic to $L_2(3) \cong A_4$. As $\text{Aut}_G(M) \triangleleft \text{Aut}_A(M)$, we get

$$\tilde{N}(M)/\tilde{C}(M) \leq N_{\text{Aut}(M)/Z(\text{Aut}(M))}(I)/I \cong Z_2.$$

By Lemma 2.1b applied to $X = N_G(M)$,

$$\tilde{C}(M) = \tilde{C}(N_G(M)).$$

Now let $\alpha \in C_A(N_G(M))$ and set $G_0 = C_G(\alpha)$. For any $u \in M^\#$, $E_u := E(C_G(u)) \cong A_5$ and $ME_u \triangleleft N_G(\langle u \rangle)$. Then $N_{E_u}(M \cap E_u) \leq N_G(M) \leq G_0$. But $C_{\text{Aut}(E_u)}(N_{E_u}(M)) = 1$, so $E_u \leq G_0$, and then $N_G(\langle u \rangle) \leq \langle E_u, M \rangle \leq G_0$. Hence either $G_0 = G$ or G_0 is strongly 5-embedded in G . Suppose the latter.

In G there is a subgroup $H \cong Sp_4(4) \cdot 2$, which we may take to contain M , by conjugation. If $H \not\leq G_0$, then $H \cap G_0$ is strongly embedded in H , contradicting the Bender-Suzuki Theorem, p. 20 of [GLS4]. So $H \leq G_0$.

Let $T \in \text{Syl}_2(G)$ with $T_0 := T \cap H \in \text{Syl}_2(H)$. Then $|T_0| = 2^9$ so $|T : T_0| = 2$. Let $S \in \text{Syl}_3(H)$. Then $C_G(v) \cong 3A_7$ and $C_H(v) \cong Z_3 \times A_5$ for all $v \in S^\#$. So $C_{\text{Aut}(C_G(v))}(C_H(v)) = 1$ for all such v , whence $C_G(v) \leq G_0$. Now $|G : G_0|$ divides $2 \cdot 7^2$ and $|G : G_0| \equiv 1 \pmod{5^2}$, so $\alpha = 1$ and

$$\tilde{C}(N_G(M)) = \tilde{1}.$$

Finally suppose that $\beta \in C_A(T)$. Now, T is isomorphic to a Sylow 2-subgroup of $L_5(2)$. Therefore T has exactly two subgroups A_1, A_2 isomorphic to E_{2^6} . Moreover, $N_i := N_G(A_i)$ satisfies $F^*(N_i) = A_i$ and $N_i/A_i \cong 3A_6$; and $H = \langle H \cap N_1, H \cap N_2 \rangle$, as $H \cap N_1$ and $H \cap N_2$ contain distinct maximal parabolic subgroups of $[H, H]$. By Lemma 2.1b,

$$\tilde{C}(T) \leq \tilde{C}(M) = \tilde{1}. \quad \square$$

Existence note: $\text{Aut}(He)$, of order $2 \cdot |He|$, is visible in a 7-local subgroup of F_1 .

G \cong **Ly** [24, 29]

$M = 3Mc2$ is the normalizer of a subgroup R of order 3 whose conjugacy class is uniquely determined by the isomorphism type of $N_G(R)$. Moreover, $\text{Aut}(M) = \text{Aut}([M, M])$ so $\text{Out}(M) = 1$ by the calculation for Mc . Therefore

$$\tilde{A} = \tilde{N}(M) = \tilde{C}(M).$$

A straightforward analysis of $M \backslash G/M$ [24] shows that there are five double cosets of uniquely determined cardinalities, and M is maximal in G . Let $\alpha \in C_A(M)$ and take any involution $t \in M$. Then α acts on $C_G(t) \cong 2A_{11}$ and centralizes $C_M(t) = (E \times R) \langle u \rangle$ where $E \cong 2A_8$ and u is an involution inverting R . Therefore $\text{Aut}(C_G(t)) \cong \Sigma_{11}$ and $C_{\text{Aut}(C_G(t))}(C_M(t)) = 1$. Consequently α centralizes $C_G(t)$. So $G = \langle M, C_G(t) \rangle \leq C_G(\alpha)$, $\alpha = 1$, and

$$\tilde{C}(M) = \tilde{1}. \quad \square$$

G \cong **Ru** [28, 27, 7]

By Sylow's Theorem $\tilde{A} = \tilde{C}(Z)$. Let $\alpha \in C_A(Z)$ set $N = M^\alpha$, and suppose that N is not G -conjugate to M , i.e., has no fixed points on G/M . In [28] this is shown to lead to a character-theoretic contradiction³ Therefore $M^\alpha \in M^G$ and

$$\tilde{A} = \tilde{N}(M).$$

Since $M \cong {}^2F_4(2)$ is complete,

$$\tilde{N}(M) = \tilde{C}(M).$$

Let $\alpha \in C_A(M)$ and $R \in \text{Syl}_3(M)$, and set $E := C_G(Z(R))$. Then $E \cong 3A_6$. But $C_{\text{Aut}(E)}(R) = 1$ so $[\alpha, E] = 1$. M is maximal in G since G has rank 3 on G/M with subdegrees 1, 1755 and 2304. Also $C_M(Z(R)) \cong SU_3(2)$ so $E \not\leq M$. Therefore $\alpha = 1$ and

$$\tilde{C}(M) = 1.$$

³A contradiction is also available by analyzing the orbit lengths of N on G/M . By construction $|MN/M| = 1755$ and there must be a second orbit of length 1755, the only possibly odd length. These orbits account for all fixed points of any 5-element of N on G/M . Remaining are 550 points on which a Sylow 5-subgroup of N acts semiregularly. N has a subgroup $N_0 \cong L_2(5^2)$, which has no such action on 550 points.

Let $\beta \in C_A(T)$. We have $|Z| = 2$ and $F^*(C) = O_2(C)$, so β centralizes C by Lemma 2.1b. Likewise there is a unique $U \triangleleft T$ with $U \cong E_{22}$, and for the same reason β acts on $N_G(U)$ as conjugation by an element of Z . Hence some $\beta' \in \{\beta, \beta z\}$ centralizes $\langle C, N_G(U) \rangle$. But $M = \langle C \cap M, N_M(U) \rangle$, these being maximal (parabolic) subgroups of M . Therefore

$$\tilde{C}(T) = \tilde{N}(M) = 1. \quad \square$$

G \cong Fi₂₂, Fi₂₃, Fi₂₄ [8, 3]

We include the nonsimple group Fi_{24} since we use the fact that $\text{Out}(Fi_{24}) = 1$ in the calculation of $\text{Aut}(Fi'_{24})$.

We have $G = Fi_n$, $n = 22, 23, 24$. Let $M(n)$ be the corresponding set of 3-transpositions. Define $Fi_{21} = U_6(2)$, also a 3-transposition group with respect to $M(21)$, the set of root involutions in Fi_{21} . We have an exact sequence $1 \rightarrow \langle t \rangle \rightarrow M \rightarrow Fi_{n-1} \rightarrow 1$ with t an involution in the 3-transposition class. Also G has rank 3 on G/M with subdegrees $3510 = 1 + 693 + 2816$, $31671 = 1 + 3510 + 28160$, $306936 = 1 + 31671 + 275264$. In particular M is maximal in G . The class of 3-transpositions is unique so

$$\tilde{A} = \tilde{N}(M), \text{ and } \tilde{N}(M)/\tilde{C}(M) \leq \text{Out}(M)$$

with $\text{Out}(M) \cong Z_2, Z_2$, and 1, respectively.

We show inductively that $C_A(M) = \langle t \rangle$. For $n = 21$, we take M as the stabilizer of a root involution, which is a maximal parabolic subgroup in $Fi_{21} \cong U_6(2)$. Thus $C_A(M) = \langle t \rangle$ when $n = 21$ (see 2.6.5e of [10]). In general let $u \in M - \langle t \rangle$ be a 3-transposition, so that $u = t^g$ for some $g \in G$. Let $\alpha \in C_A(M)$. Then α centralizes u , acts on M^g and centralizes $M \cap M^g = C_{M^g}(t)$. By induction, α acts on $M^g / \langle u \rangle$ like an element of $\langle t \rangle$. Consequently α centralizes $O^2(M^g)$. The same holds for αt , which also centralizes M . Therefore some element of $\alpha \langle t \rangle$ centralizes $\langle M, O^2(M^g) \rangle = G$, so $\alpha \in \langle t \rangle$ as claimed. In particular

$$\tilde{C}(M) = 1.$$

Similarly we show that $C_A(T) \leq G$. Indeed let $\beta \in C_A(T)$. Then β centralizes t , acts on M , and inductively induces an inner automorphism on $M / \langle t \rangle$. Hence for some $x \in M$, $\beta' := \beta x$ centralizes $M / \langle t \rangle$, so centralizes $[M, M] \langle t \rangle = M$. Thus $\beta \in \beta' G = G$, and

$$\tilde{C}(T) = 1.$$

Now if $G = Fi_{23}$, then G has there is also a unique class of involutions v such that $N := C_G(v) \cong 2^2 U_6(2).2$. Observe that v is 2-central in G . Then $\tilde{A} = \tilde{N}(N)$, $\tilde{N}(N) = \tilde{C}(N)$ since $\text{Out}(N) = 1$, and $\tilde{C}(N) \leq \tilde{C}(T) = 1$. We conclude that $|\tilde{A}| \leq 2$ for $n = 22$ and $\tilde{A} = 1$ for $n = 23, 24$. \square

Existence note: There is an involution $y \in Fi'_{24}$ such that $C_{Fi'_{24}}(y)/\langle y \rangle$ is an extension of Fi_{22} by a non-inner automorphism.

$G \cong Fi'_{24}$ [8]

Since $|Fi_{24} : Fi'_{24}| = 2$ and $Z(Fi_{24}) = 1$, the image \tilde{A}_0 of Fi_{24} in \tilde{A} is a group of order 2. Since Fi_{24} is complete, as we have just seen,

$$\tilde{A}_0 = N_{\tilde{A}}(\tilde{A}_0).$$

Thus $|\tilde{A}|/2$ is odd. Let $\alpha \in A$ with \tilde{A} of odd order, and set $A_1 = G\langle\alpha\rangle$. We show that $\tilde{A}_1 = 1$. By Sylow's Theorem,

$$\tilde{A}_1 = (N_{A_1}(C))^\sim.$$

Let $Q = F^*(C)$. Then $Q \cong 2_+^{1+12}$ and $F^*(C/Q) \cong 3U_4(3)$. Then $C/Q \leq N_{\text{Aut}(Q)}(O_3(C/Q)) \cong \Gamma U_6(2)$, the product of $GU_6(2)$ with a graph automorphism. The corresponding 6-dimensional representation of $3U_4(3)$ over \mathbf{F}_4 is absolutely irreducible, so C/Q is self-centralizing in $\text{Aut}(Q)$. As $\text{Out}(U_4(3))$ is a 2-group and \tilde{A}_1 has odd order, $\text{Aut}_G(Q)$ is self-normalizing in $\text{Aut}_{A_1}(Q)$. Therefore

$$(N_{A_1}(C))^\sim = (C_{A_1}(C))^\sim = (C_{A_1}(T))^\sim$$

as in Lemma 2.1e. We may therefore assume that $[\alpha, T] = 1$.

Choose a non-2-central involution $t \in T$ such that t is extremal in T . Then $F := F^*(C_G(t)) \cong 2Fi_{22}$ and α maps into $O^2(C_{\text{Aut}(F)}(T \cap F))$. As $T \cap F \in \text{Syl}_2(F)$ we conclude from the Fi_{22} -calculation that $[\alpha, F] = 1$ and then as $|C_G(t) : F| = 2$, $[\alpha, C_G(t)] = 1$.

Now $M \cong Fi_{23}$ with $T_M \in \text{Syl}_2(M)$. We have $C_{Fi_{24}}(M) = \langle z \rangle \cong Z_2$ with $z \notin Fi'_{24}$. Also, there exist involutions $t, u \in Z(T_M)$ such that $C_M(t) \cong 2Fi_{22}$ and $C_M(u) \cong 2^2U_6(2) \cdot 2$. Then α centralizes $\langle C_M(t), C_M(u) \rangle$, which equals M as $C_M(t)$ is maximal in M . The suborbits of M on G/M are the same as those of $N_{Fi_{24}}(M) \cong Z_2 \times M$ on $Fi_{24}/N_{Fi_{24}}(M)$, and in particular M is maximal in G . As $T \not\leq M$, it follows that $\alpha = 1$ and

$$\tilde{C}(T) = (C_{A_1}(C))^\sim = \tilde{1}. \quad \square$$

$G \cong \mathbf{F}_5$ [14, 15]

There exists a unique conjugacy class C_3 of elements $g \in G$ of order 3 such that $C_G(g) = \langle g \rangle \times E(C_G(g))$, with $E(C_G(g)) \cong A_9$, and $N_G(\langle g \rangle) = C_G(g)\langle u \rangle$, with $E(C_G(g))\langle u \rangle \cong \Sigma_9$. There is an element $g_1 \in C_3$ such that $C_G(g) \leq M$. Choose three disjoint 3-cycles $g_2, g_3, g_4 \in E(C_G(g_1))$. Then $A = GN_A(\langle g_1 \rangle) = GN$, where N stabilizes the set of four subgroups $\langle g_i \rangle$, $i = 1, \dots, 4$ and so normalizes $M = \langle E(C_G(g_i)) \mid 1 \leq i \leq 4 \rangle$. Thus

$$\tilde{A} = \tilde{N}(M) \text{ and } \tilde{N}(M)/\tilde{C}(M) \leq \text{Out}(M) \cong Z_2.$$

Let $\alpha \in C_A(M)$ and put $C = C_G(\alpha)$. Let $f \in M$ be a 5-cycle. Then α centralizes $C_M(f) = \langle f \rangle \times H$, $H \cong A_7$. From [6] or Table 5.3w of [10], $F^*(C_G(f)) = \langle f \rangle \times I$, $I \cong U_3(5)$. Thus α maps into $C_{\text{Aut}(I)}(H)$, which, however, is trivial⁴. Hence $C \geq \langle M, H \rangle$. This implies that $|G : C|$ divides $2^5 \cdot 3 \cdot 5^2 \cdot 19$.

Now C , like M , contains a subgroup E of order 55. Then $O_2(C_G(E)) \cong Z_2$ is α -invariant and so lies in C . Hence a Sylow 11-normalizer in C has index 1 or 2 in a Sylow 11-normalizer in G . Consequently $|G : C| \equiv 1$ or $2 \pmod{11}$. Likewise M contains a Sylow 7-normalizer of G and so $|G : C| \equiv 1 \pmod{7}$.

If α is a 5'-element then it acts on $N_G(Z(R))$, where $R \in \text{Syl}_5(I)$. But $Z(R)$ is a Sylow 5-center in G and $F^*(N_G(Z(R))) \cong 5^{1+4}$ contains $\langle f \rangle \times R$ with index 5. Hence α must centralize $F^*(N_G(Z(R)))$ and then α centralizes $N_G(Z(R))$, by Lemma 2.1a. Thus in this case $|G : C| \equiv 1 \pmod{5}$ as well, and $|G : C|$ divides $2^5 \cdot 3 \cdot 19$. There are no such numbers $|G : C| > 1$.

Therefore we may assume that α is a 5-element. Expand $Q \in \text{Syl}_3(M)$ to $P \in \text{Syl}_3(G)$, and let $N = N_G(Z(P))$. Then $|P : Q| = 3$ and $F^*(N) \cong 3^{1+4}$. Hence α centralizes $F^*(N)$ and then N , by Lemma 2.1a. Now $|G : C|$ divides $2^5 \cdot 5^2 \cdot 19$ and $|G : C| \equiv 1 \pmod{21}$, and $|G : C| \equiv 1$ or $2 \pmod{11}$. Again this forces $|G : C| = 1$, so

$$\tilde{C}(M) = 1. \quad \square$$

Existence note: The automorphism group $A = F_5.2$ is involved in $N_{F_1}(D)$, where D is a subgroup of F_1 of order 5 and class 5A [6] or 5.3z in [10].

G \cong F₃ [33]

Since $M = C$ and T has one class of involutions,

$$\tilde{A} = \tilde{N}(M).$$

There are subgroups $E \leq D = N_G(E)$ with $E = C_G(E) \cong E_{2^5}$ and $D/E \cong \text{Aut}(E)$.

Then D contains a Sylow 2-subgroup of G and by replacing D by a suitable conjugate we may assume that $Z \in E$, $D \cap M/E \cong 2^4.L_4(2)$ and $D \cap M/O_2(M) \cong A_8 \cong L_4(2)$.

We have $F = O_2(M) \cong 2_+^{1+8}$. By Lemma 2.1d, the restriction mapping $\text{Aut}(M) \rightarrow \text{Aut}(F)$ is injective. Furthermore, since A_9 contains a Frobenius group of order 9.8, M acts absolutely irreducibly on F/Z . Therefore $\text{Aut}(M)/\text{Aut}_F(M)$ embeds in $\text{Aut}(M/F) \cong \Sigma_9$. However, for any $x \in M$ of order 3 mapping onto a 3-cycle $\bar{x} \in \bar{M} := M/F \cong A_9$, we have $C_F(x) = Z$. If $\text{Aut}(M)/\text{Aut}_F(M) \cong \Sigma_9$, then \bar{M} contains a subgroup $\langle \bar{t} \rangle \times \bar{H}$ with \bar{t} an involution inverting \bar{x} and $\bar{H} \cong \Sigma_7$. Since \bar{x} is fixed-point-free on F/Z , $\langle \bar{t} \rangle$ is free on

⁴One way to see this is to observe by groups orders that $I = HB$, where B is a Borel subgroup of I containing some Sylow 5-subgroup of H . Note that Sylow 2-subgroups of B are cyclic so $|HB|_2 > |B|_2$. Since α centralizes a 5-element of B , α normalizes B . But then $B \geq [\alpha, B] = [\alpha, HB] = [\alpha, I] \triangleleft I$, so $[\alpha, I] = 1$.

F/Z and by the $A \times B$ -lemma, \overline{H} acts faithfully on $C_{F/Z}(\bar{t}) \cong E_{2^4}$, which is impossible as Σ_7 does not embed in $L_4(2) \cong A_8$. Therefore $\text{Aut}(M)/\text{Aut}_F(M) \cong A_9$ and

$$\tilde{N}(M) = \tilde{C}(M).$$

We set $N := \langle C, D \rangle$ and argue that $C_G(u) \leq N$ for all involutions $u \in N$, whence $N = G$ by the Bender-Suzuki Theorem (p. 20 of [11]). Since $|G : C \cap D|$ is odd we may assume that $u \in C \cap D$. Suppose first that $u \notin F$. Then in $\overline{C} = C/F \cong A_9$, \overline{u} inverts 3-cycle, which is fixed-point-free on F/Z , so u acts freely on F/Z . If \overline{u} is a root involution then for some element $v \in C$ of order 5, $[\overline{v}, \overline{u}] = 1$ and $C_G(v)/O_5(C_G(v)) \cong SL_2(3)$. Hence $C_G(v)$ contains an element w of order 4 such that $\langle w^2 \rangle = Z$ and $\overline{w} = \overline{u}$. By the free action of u on F/Z , u is conjugate to an element of $\langle w \rangle$, which is absurd as u is an involution. The mapping $C/F \rightarrow \text{Aut}(\overline{E})$ is an isomorphism, so \overline{u} is a 2-central involution in $\overline{C} \cong A_9$. Now \overline{u} is a transvection on $\overline{E} \cong E_{2^4}$. On the other hand \overline{u} acts freely on \overline{F} , as we saw above. Therefore $|\langle \overline{u}, \overline{F} \rangle \cap \overline{E}| = 8$. But u centralizes $[u, F] \cap E$. It follows that u induces a transvection on E . Therefore u is D -conjugate to an element of $O_2(D \cap E) = F$.

As $u^N \cap F \neq \emptyset$, we now may assume that $u \in F$. Since $C \cap D/F \cong L_4(2)$ has a natural module and its dual as composition factors on F/Z , $C \cap D$ has two orbits on the set of involutions in $F - Z$: those in E , and the rest. But C is irreducible on \overline{F} and hence transitive on the involutions of $F - Z$. As D is transitive on $E^\#$, it follows that N has one class of involutions; and then as $C \leq N$ we conclude that $N = G$, as asserted.

Finally let $\alpha \in C_A(M)$. Then α centralizes $C \cap D$ so normalizes D . By Lemma 2.1b, $\alpha \downarrow D \in \text{Aut}_Z(D)$. Hence for some $\beta \in \alpha Z$, $C_G(\beta) \geq \langle C, D \rangle = N = G$. Therefore $\tilde{C}(M) = 1$.

G \cong F₂ [6]

Here $M = C_G(t)$, $F^*(M) \cong 2^2 E_6(2)$, and $|M : F^*(M)| = 2$; moreover there exists an involution $u \in M - F^*(M)$ such that $C_{F^*(M)}(u) \cong F_4(2) \times Z_2$. Only the involutions in t^G have centralizers isomorphic to M , so

$$\tilde{A} = \tilde{N}(M).$$

Since $\text{Out}(2^2 E_6(2)) \cong \Sigma_3$ [10, Sec. 2.5], $M/\langle t \rangle$ is perfect.

$$\tilde{N}(M) = \tilde{C}([M, M]/\langle t \rangle).$$

Let $\alpha \in C_A([M, M]/\langle t \rangle)$. We argue first that $[\alpha, M] = 1$ and then that $[\alpha, G] = 1$. Since $M/\langle t \rangle$ is centerless and $[M, M]$ is quasisimple, α centralizes $M/\langle t \rangle$ and $[M, M]$. If α does not centralize M , therefore, we must have $u^\alpha = tu \in u^G$. However, from [10, Table 5.3y] we see that of the involutions u and tu , one lies in the class t^G and the other does not, belonging instead to class $2C$. This contradiction shows that

$$\tilde{N}(M) = \tilde{C}(M).$$

It also shows that we may take $u \in t^G$. Now let $\alpha \in C(M)$. There is $v \in M$ of order 3 such that $N := N_{M/\langle t \rangle}(\langle v \rangle) \cong \Sigma_3 \times U_6(2) \cdot 2$ [10, Table 7.3.4]. Having a nonsolvable centralizer, v must belong to class 3A [10, Table 5.3y], and so $J := N_G(\langle v \rangle) \cong \Sigma_3 \times \text{Aut}(Fi_{22})$. Therefore α acts on $E(J) \cong Fi_{22}$ and centralizes both t and $E(C_{E(J)}(t)) = N^{(\infty)} \cong U_6(2)$. In the discussion of the case $G = Fi_{22}$, we have argued that (in the terminology of the current case) $C_{\text{Aut}(E(J))}(N^{(\infty)} \langle t \rangle) = \langle t \rangle$. Therefore replacing α by αt if necessary, we may assume that $[\alpha, E(J)] = 1$. Now $t^{E(J)}$ is a class of 3-transpositions in $E(J)$. Hence there is $g \in E(J)$ such that tg has order 3 and $E(C_{E(J)}(tg)) \neq 1$. In particular $C_G(tg)$ is nonsolvable, so $N_G(\langle tg \rangle) \cong S_3 \times \text{Aut}(Fi_{22})$ [10, Table 5.3y].

Finally consider $H := C_G(\alpha)$. We have $M \leq H$ and $g \in E(J) \leq H$. Then

$$\begin{aligned} M \cap M^g &= C_G(\langle t, t^g \rangle) \cong \text{Aut}(Fi_{22}), \\ |H| &\geq |MM^g| = \frac{|M|^2}{|M \cap M^g|} = \frac{(4|E_6(2)|)^2}{2|Fi_{22}|}, \\ |G : H| &\leq \frac{|F_2||Fi_{22}|}{8|E_6(2)|^2} = \frac{3^4 5^4 23 \cdot 31 \cdot 47}{2^{17} 7 \cdot 17 \cdot 19} \leq 6. \end{aligned}$$

As G is simple and $|G| > 6!$, $G = H$, and so

$$\tilde{C}(M) = 1. \quad \square$$

$\mathbf{G} \cong \mathbf{F}_1 \quad [12]$

Here $M = C$. By Sylow's Theorem,

$$\tilde{A} = \tilde{N}(M).$$

The action $C/F^*(C) \cong Co_1$ on $F^*(C)/\langle z \rangle \cong \Lambda/2\Lambda$ is absolutely irreducible, and $H^1(C/F^*(C), \Lambda/2\Lambda)$ is trivial by Corollary 2.6. Therefore by Lemma 2.1b,

$$\tilde{N}(M) = \tilde{C}(M).$$

In particular $\tilde{C}(T) \leq \tilde{N}(M) = \tilde{C}(M)$.

Let $\alpha \in C(M)$. Let $t \in M$ be a non-2-central involution which is extremal in T and set $H := C_G(t)$. Then α acts on $H \cong 2F_2$, and α centralizes $H \cap T = C_T(t) \in \text{Syl}_2(H)$. By the F_2 case, $[C_T(t), \alpha] = 1$ implies that $[H, \alpha] = 1$. Hence $C_G(\alpha)$ contains M and H .

But it is well-known, and we give an elementary proof below, that

$$\langle M, H \rangle = G. \quad (3A)$$

Thus α will necessarily be trivial and we will have proved

$$\tilde{C}(M) = 1.$$

Our argument that $\langle M, H \rangle = G$ begins by setting $G_0 = \langle M, H \rangle$ and observing that $F^*(C_G(\langle z, t' \rangle)) = F^*(C_M(t))$ is a 2-group for any involution t' such that $[z, t'] = 1$. Taking $t' = t^g$ we have $[z^{g^{-1}}, t] = 1$ and $F^*(C_G(\langle z^{g^{-1}}, t \rangle))$ is a 2-group. As g varies, $z^{g^{-1}}$ varies over all of $z^G \cap C_G(t) = z^G \cap H$. Therefore any $z' \in z^G \cap H$ is H -conjugate to z_1 or z_2 , these being conjugacy class representatives such that $C_{H/\langle t \rangle}(z_1) \cong 2^{1+22}Co_2$ and $C_{H/\langle t \rangle}(z_2) \cong 2^{9+16}O_8^+(2)$. To pull back to H , notice that we may take $z_1 = z$ in the first case, and then $t \in O_2(C)$ so t and tz are C -conjugate. In the second case, writing $z_2 = z^{g^{-1}}$ and $t' = t^g \in C$, we have $C_C(t') = C_G(\langle t', z \rangle) = C_G(\langle t, z_2 \rangle^g) \cong C_G(\langle t, z_2 \rangle) = C_H(z_2)$, whence $t' \notin O_2(C)$ and the image of t' in $C/O_2(C) \cong Co_1$ is a 2-central involution, with $C_{C/O_2(C)}(t'O_2(C))$ is an extension of a 2-group by just $\Omega_8^+(2)$. Consequently z_2 and z_2t are fused in H . We have proved that there are exactly two G -orbits on the set P_G of all pairs (z^h, t^k) such that $h, k \in G$ and $[z^h, t^k] = 1$, and they are represented by (z, t) and (z, t') .

Since $C \leq G_0$, it follows that there are exactly one or two G_0 -orbits on the set P_H of all pairs (z^h, t^k) as above but with h, k restricted to lie in G_0 ; and representatives of these orbits are (z, t) and (in the two-orbit case) (z, t') . We consider these cases separately.

In the one-orbit case, $t' \notin t^{G_0}$ and $t^G \cap C \subseteq O_2(C)$. Therefore $t^G \cap (C \cap H) \subseteq O_2(C) \cap H = O_2(C \cap H)$. Since $O_2(C \cap H)/\langle z, t \rangle \cong E_{2^{22}}$ is acted on irreducibly by $C \cap H/O_2(C \cap H) \cong Co_2$, and $t^C \not\leq \langle t, z \rangle$, it follows that $\langle t^{G_0} \cap C \cap H \rangle = O_2(C \cap H)$. On the other hand, by the structure of $H/Z(H) \cong F_2$, there is $t^* \in t^G \cap H$ such that in $C_G(\langle t, t^* \rangle) \cong 2^{22}E_6(2)$, long root involutions are H -conjugate to t . The positive long root subgroups generate a 2-group isomorphic to a Sylow 2-subgroup of $D_4(2)$, which is not embeddable in the class 2 group $O_2(C \cap H)$, a contradiction.

Therefore we must be in the two-orbit case, so $t^G \cap C = t^{G_0} \cap C$, that is, $t^G \cap G_0 = t^{G_0}$. From this one quickly gets $t^x \in G_0 \iff x \in G_0$ for all $x \in G$, and one could argue that for any involution $y \in G_0$, $y^x \in G_0 \iff x \in G_0$. This would give $G_0 = G$, as desired, since G_0 cannot be strongly embedded in G by the Bender-Suzuki Theorem. We argue differently, however. Using [10, Table 5.3], we see that in $C/O_2(C) \cong Co_1$ there exists a subgroup $D \times W$ with $D \cong D_{10}$ and $W \cong A_5 \wr Z_2$; moreover, the only involutions centralizing an isomorphic copy of W are 2-central. We have seen that there is a conjugate $t' \in t^G$ such that $t' \in C$ and the image of t' in $C/O_2(C)$ is 2-central. Hence using the Baer-Suzuki theorem, t' inverts an element of G_0 of order 5. Moreover, $t' \in G_0 \cap t^G = t^{G_0}$. Thus, t inverts some $f \in G_0$ of order 5. Let $t'' = tf$. Then in the action of G on $t^G \times t^G$, the stabilizer of (t, t'') is $G_{t, t''} = C_G(f, t)$. From [10, Table 5.3z] we see that $|G_{t, t''}| \leq |F_5| = 2^{14}3^65^67.11.19$. Note that all G -conjugate pairs of the form (t, t^*) are actually $H = C_G(t)$ -conjugate. Thus,

$$\begin{aligned} |G_0 : H| &= |t^{G_0}| \geq |\{t^* \in t^{G_0} \mid (t, t^*) \in (t, t'')^G\}| \geq |H|/|G_{t, t''}| \geq |H|/|F_5|, \\ |G : G_0| &\leq \frac{|G||F_5|}{|H|^2} = \frac{5^3 7^3 \cdot 11 \cdot 13 \cdot 29 \cdot 41 \cdot 59 \cdot 71}{2^{24} \cdot 17 \cdot 23 \cdot 31 \cdot 47} < 4. \end{aligned}$$

As G is simple, $G = G_0$, as desired.

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